Swami Vivekananda School of Engg & Tech

Note - Control System

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DIFFERENT EQUATION OF PHYSICAL SYSTEM

Physical system :-

It is a collection of physical objects connected together to serve an objective.

Ex → The governing mechanism of a steam turbine (or a communicating satellite orbiting the earth.

SYSTEM :-

It is used to describe a combination of components which may not all be physical.

Ex → Biological, economic, socio-economic system.

Note : physical system can not be represents it's full physical properties, so an ideal assumption is required for the purpose of analysis and synthesis of the system.

→ An idealised physical system is called a physical model

→ once a physical model of a physical system is obtained, the next step is to obtain a mathematical model which is the mathematical representation of a physical model through use of appropriate physical laws.

→ When the mathematical model of a physical system is solved for various input conditions, the result represents the dynamic response of the system.

Ex → The mathematical model of a system is linear, If it obeys the principle of superposition and homogeneity.

If $x_1(t), x_2(t)$ → i/p.

$y_1(t), y_2(t)$ → o/p.

Then linear combination of these input and o/p are

$$\alpha_1 x_1(t) + \alpha_2 (x_2(t)) = \alpha_1 y_1(t) + \alpha_2 y_2(t).$$

Where α_1, α_2 → Constant.

Through variable V_T –

Which passes through the element and has the same value in at one port and out of the other.

Ex → Current through electrical resistance.

Across variable V_A – Appears across the two terminals of the elements.

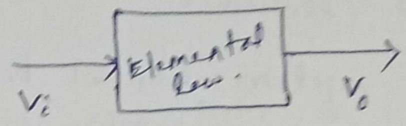
Ex → Voltage across electrical resistance.

I/p variable – Independent variable (V_i).

O/p variable → Dependent variable (V_o).

$V_i \rightarrow V_A$

$V_o \rightarrow V_T$.



$V_o = f(V_i)$.

(Block diagram of elements)

Mechanical system –

Translatory → Being uniform motion in one direction.

Torsional → The twisting of a bodily organ (or) part of its own axis.

Mass : $E = \frac{1}{2} m v^2$

Inertia = $E = \frac{1}{2} J \omega^2$.

Spring (translatory) = $E = \frac{1}{2} K x^2$.

Spring (torsional) = $E = \frac{1}{2} K \theta^2$.

$P = f v^2$ (tw)

$= f \omega^2$ (tw)

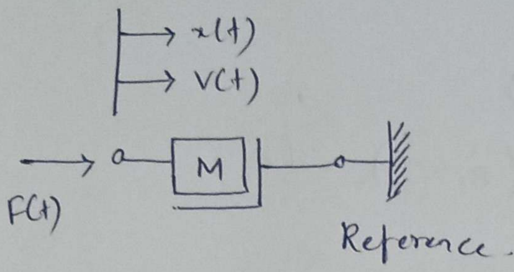
$$\int_{-\infty}^{t} F\,dt = M \int_{-\infty}^{t} \left(\frac{dv}{dt}\right) dt.$$

$\Rightarrow P = Mv$ if $v(-\infty) = 0$.

Here f = viscous friction coefficient

θ = Angular displacement

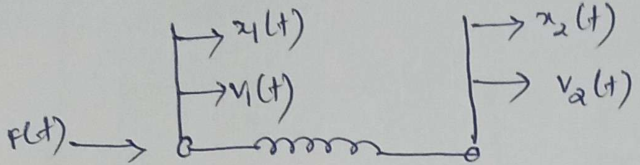
① The mass element.



$$F = M\frac{dv}{dt} = M\frac{d^2x}{dt^2}.$$

x = displacement.

② The spring element



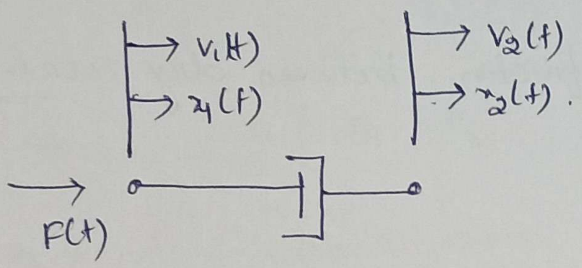
K = ~~Spring~~ Torsional spring coefficient.

$$F = K(x_1 - x_2) = Kx$$

$$= K\int_{-\infty}^{t}(v_1 - v_2)\,dt = K\int_{-\infty}^{t} v\,dt$$

$\therefore K =$

③ The damper element

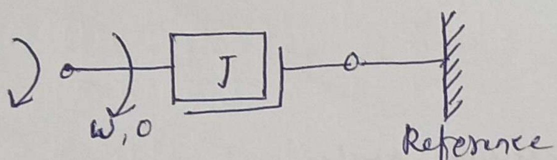


$$F = f(v_1 - v_2).$$

$$= fv = f(\dot{x}_1 - \dot{x}_2) = f\dot{x}.$$

Rotational elements

④ The inertia element :

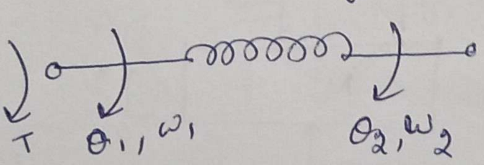


$$T = J\frac{d\omega}{dt} = J\frac{d^2\theta}{dt^2}.$$

ω = velocity.

θ = Displacement.

⑤ The torsional spring element ~~constant~~ :

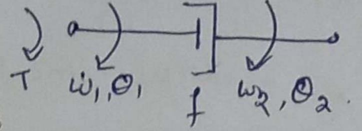


$$T = K(\theta_1 - \theta_2).$$

$$= K\theta = K\int_{-\infty}^{t}(\omega_1 - \omega_2)\,dt$$

$$= K\int_{-\infty}^{t}\omega\,dt$$

⑥ The damper element.



$$T = f(\omega_1 - \omega_2)$$
$$= f\omega = f(\dot{\theta}_1 - \dot{\theta}_2) = f\dot{\theta}$$

Friction

Whenever mechanical surface are operated in sliding contact then friction exists.

Friction
- Coulomb friction force
- Viscous friction force
- Stiction.

Coulomb friction force :-

The force of sliding friction between dry surface.

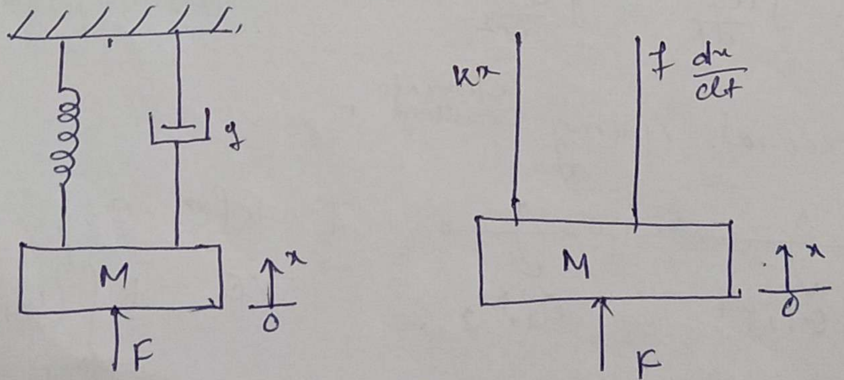
→ Force is generally const.

Viscous friction force :-

Friction betn moving surface separated by fluid.
(or) Force between solid body and fluid medium.

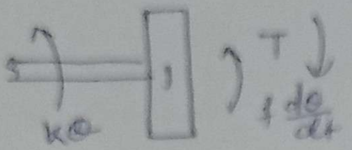
Stiction - The force required to intiate motion between two contacting surfaces.

A Mass - spring dashpot system.



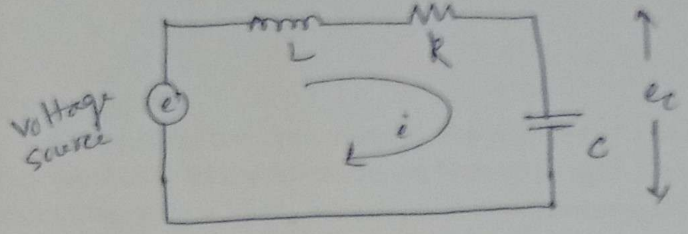
$$F - f\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2} \Rightarrow F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + kx.$$

Rotational system



$$T = J\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} + K\theta$$

Electrical system



$$e = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt$$

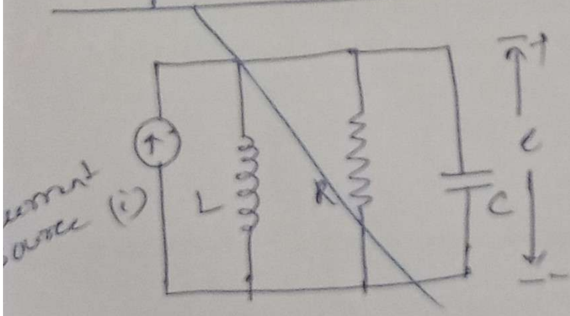
$$\frac{1}{C}\int i\,dt = e_c$$

In terms of electrical charge

$$q = \int i\,dt$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e$$

For parallel circuit



$$i = C\frac{de}{dt} + \frac{1}{L}\int_{-\infty}^{t} e\,dt + \frac{e}{R}$$

In terms of magnetic flux linkage $\phi = \int e\,dt$

$$C\frac{d^2\phi}{dt^2} + \frac{1}{L}\phi + \frac{1}{R}\frac{d\phi}{dt} = i$$

Lecture-6
Derivation of Transfer functions, Block Diagram Algebra.

The system can be represented in two forms:

- Block diagram representation
- Signal flow graph

Block diagram

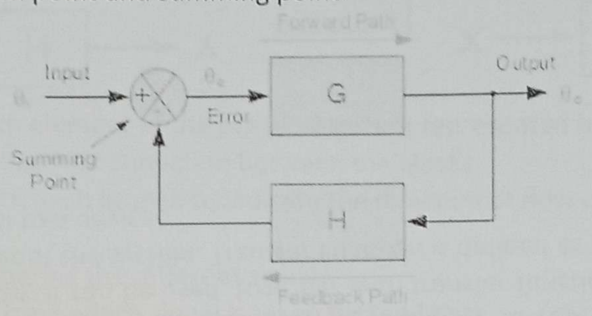
A pictorial representation of the functions performed by each component and of the flow of signals

Basic elements of a block diagram

- Blocks
- Transfer functions of elements inside the blocks
- Summing points
- Take off points
- Arrow

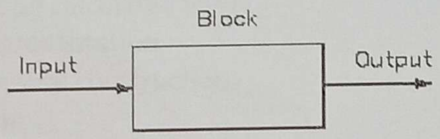
Block diagram

A control system may consist of a number of components. A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals. The elements of a block diagram are block, branch point and summing point.



Block

In a block diagram all system variables are linked to each other through functional blocks. The functional block or simply block is a symbol for the mathematical operation on the input signal to the block that produces the output.

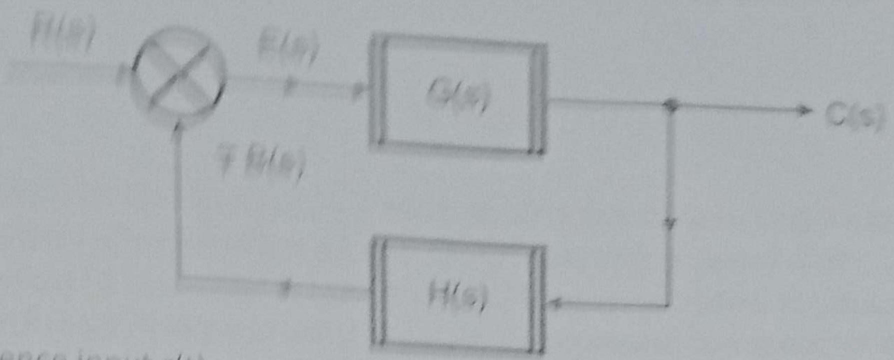


Summing point

Although blocks are used to identify many types of mathematical operations, operations of addition and subtraction are represented by a circle, called a summing point. As shown in Figure a summing point may have one or several inputs. Each input has its own appropriate plus or minus sign.

A summing point has only one output and is equal to the algebraic sum of the inputs.

...onical form of closed loop system



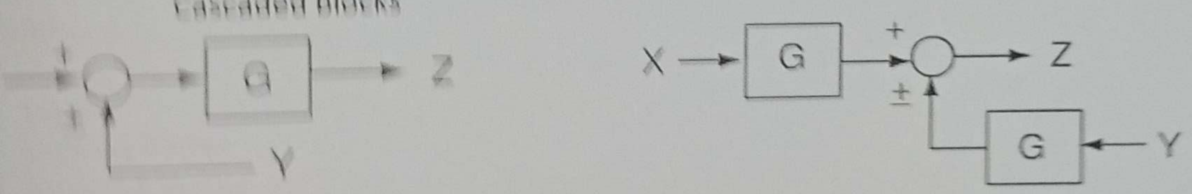
...of reference input r(t)
...of controlled output c(t)
...of error signal e(t)
...of feedback signal b(t)
...d path transfer function
...ck path transfer function

...n reduction technique

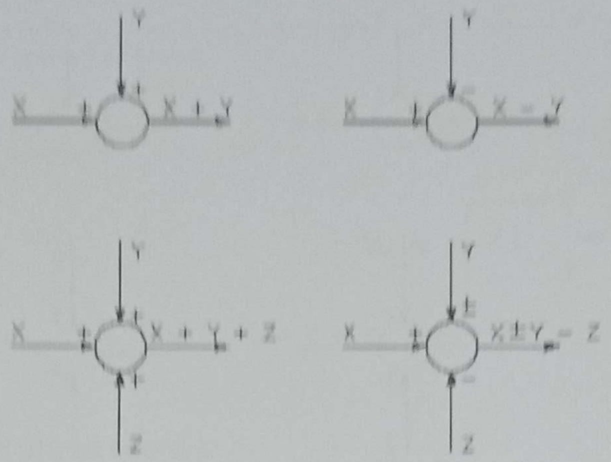
...of their simplicity and versatility, block diagrams are often used by control engineers to ...types of systems. A block diagram can be used simply to represent the composition and ...ion of a system. Also, it can be used, together with transfer functions, to represent the cause-...lationships throughout the system. Transfer Function is defined as the relationship between an ...and an output signal to a device

...m rules

Cascaded blocks

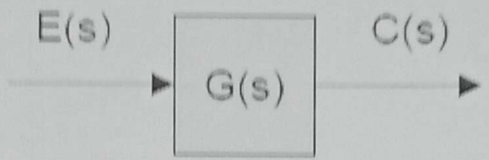


Moving a summer beyond the block moving



A takeoff point is used to allow a signal to be used by more than one block or summing point. The
ansfer function is given inside the block

- The input in this case is E(s)
- The output in this case is C(s)
- C(s) = G(s) E(s)



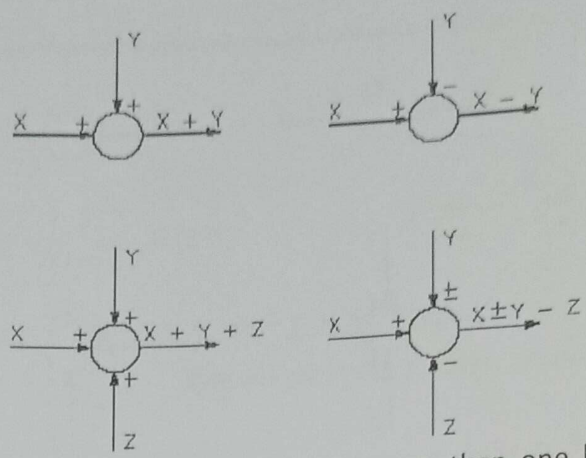
- **Functional block** – each element of the practical system represented by block with its
- **Branches** – lines showing the connection between the blocks
- **Arrow** – associated with each branch to indicate the direction of flow of signal
- **Closed loop system**
- **Summing point** – comparing the different signals
- **Take off point** – point from which signal is taken for feed back

vantages of Block Diagram Representation

- Very simple to construct block diagram for a complicated system
- Function of individual element can be visualized
- Individual & Overall performance can be studied
- Over all transfer function can be calculated easily

advantages of Block Diagram Representation

- No information about the physical construction
- Source of energy is not shown



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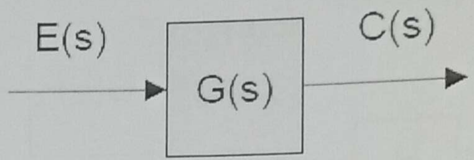

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Disadvantages of Block Diagram Representation

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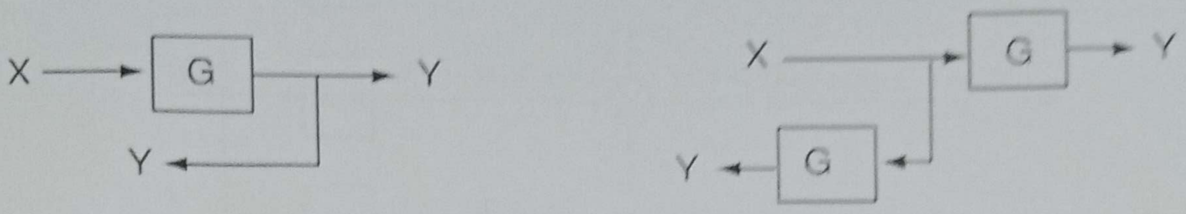
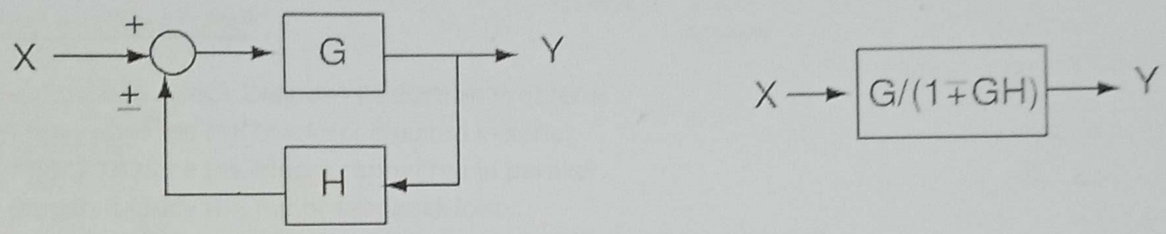
Moving a summer ahead of block

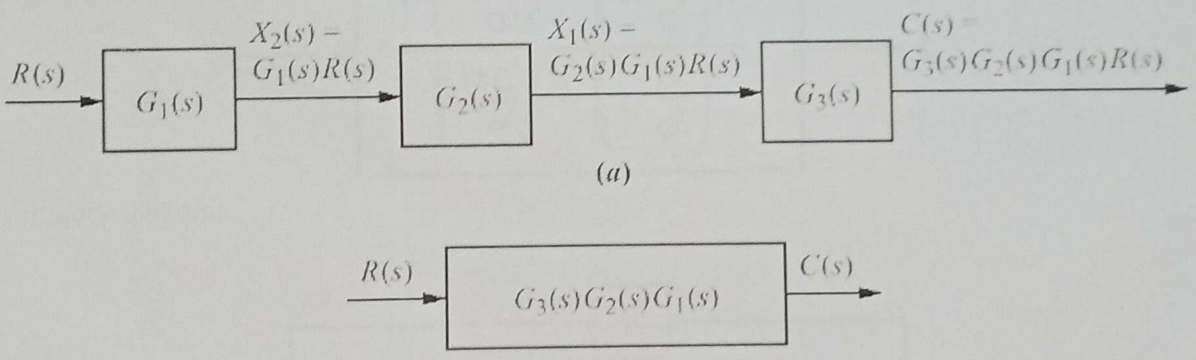


Moving a pick-off ahead of block

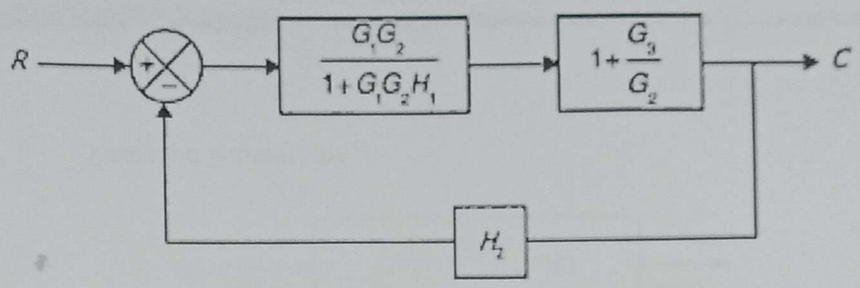
Moving a pick-off beyond a block



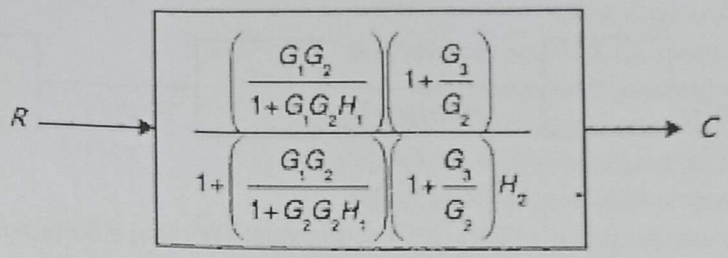
Eliminating a feedback loop



Cascaded Subsystems



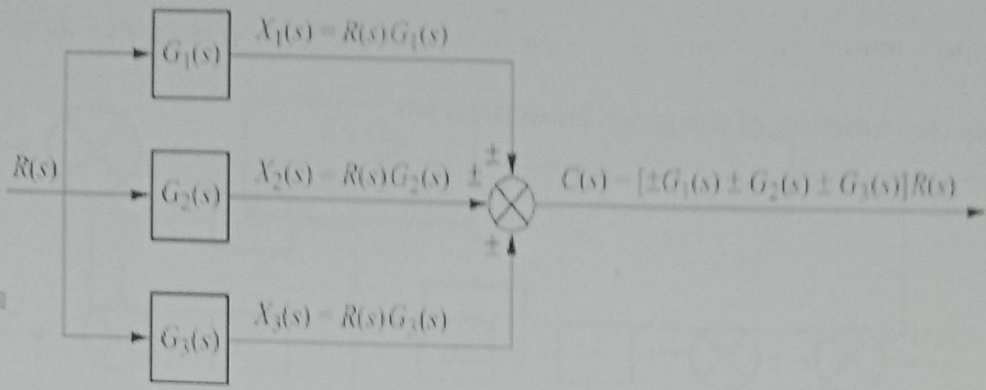
Equivalent block diagram



Transfer function

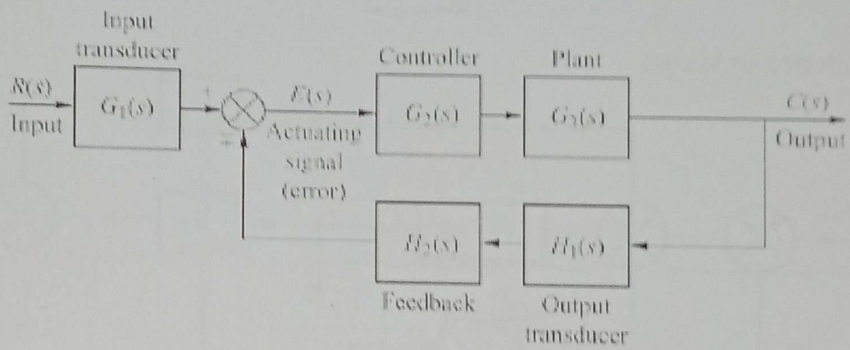
$$\frac{C}{R} = \frac{\dfrac{G_1(G_2+G_3)}{1+G_1G_2H_1}}{1+\dfrac{G_1(G_2+G_3)H_2}{1+G_1G_2H_1}}$$

$$\frac{G_1(G_2+G_3)}{1+G_1G_2(H_1+H_2)+G_1G_3H_2}$$



Parallel Subsystems

Feedback Control System

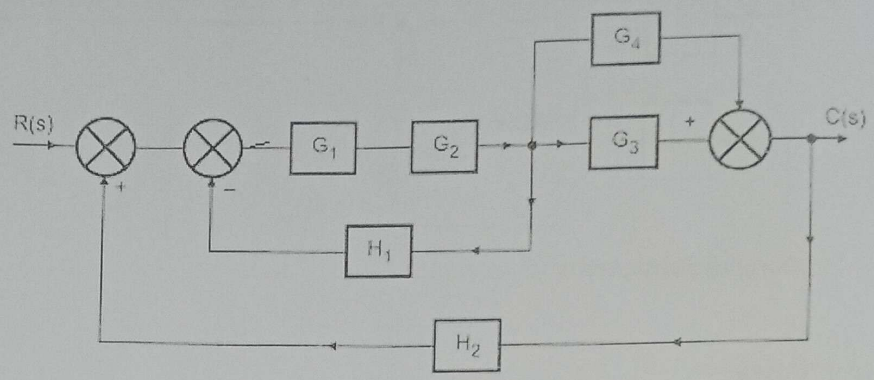


Procedure to solve Block Diagram Reduction Problems

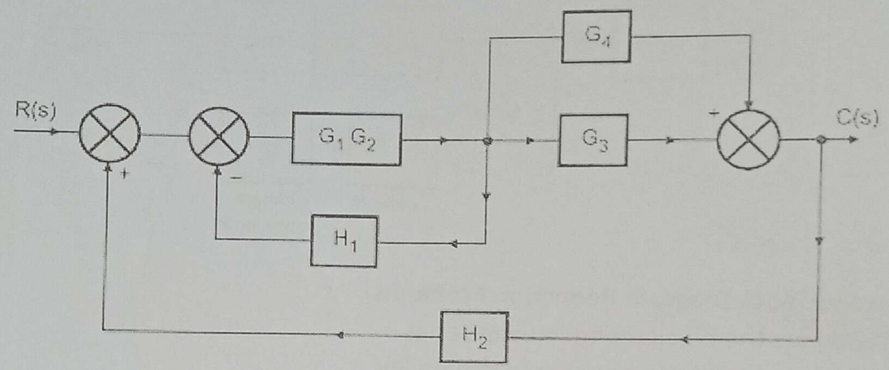
- Step 1: Reduce the blocks connected in series
- Step 2: Reduce the blocks connected in parallel
- Step 3: Reduce the minor feedback loops
- Step 4: Try to shift take off points towards right and Summing point towards left
- Step 5: Repeat steps 1 to 4 till simple form is obtained
- Step 6: Obtain the Transfer Function of Overall System

Problem 1

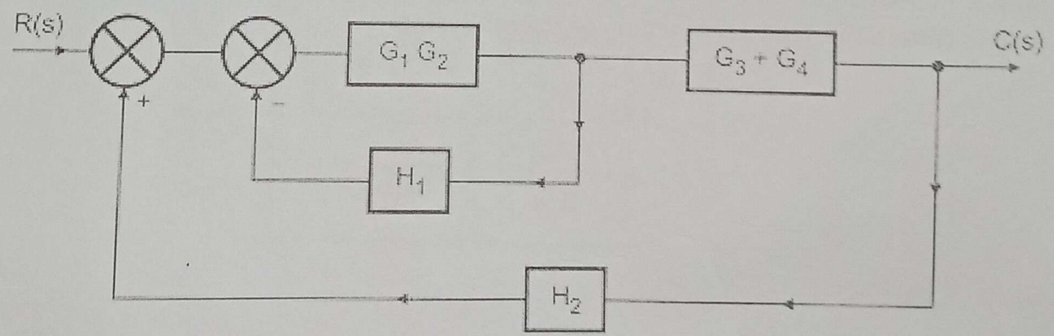
Obtain the Transfer function of the given block diagram



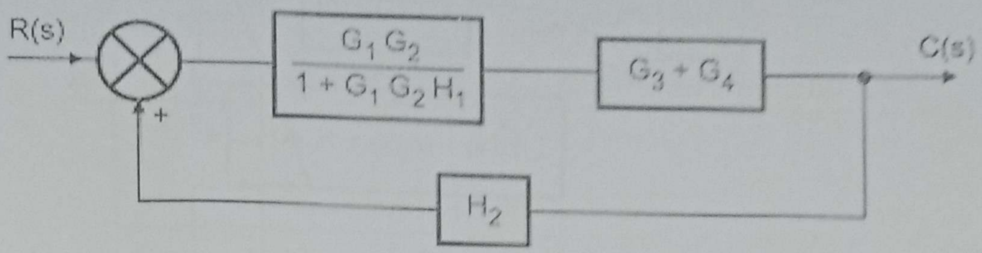
Combine G1, G2 which are in series

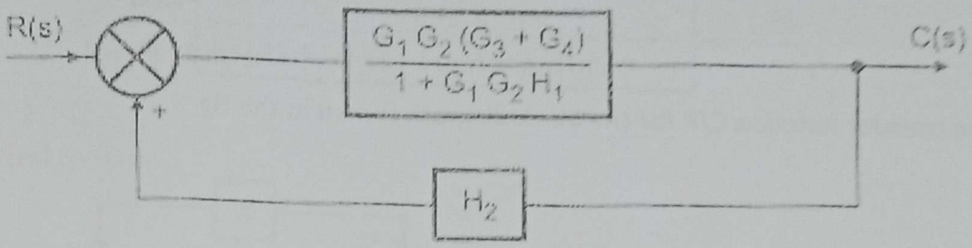


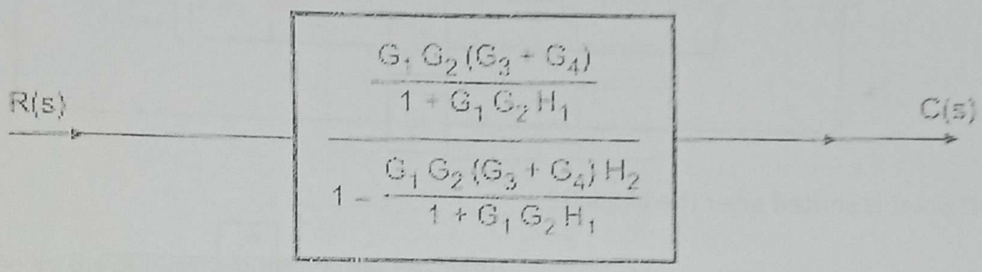
Combine G3, G4 which are in Parallel



Reduce minor feedback loop of G1, G2 and H1



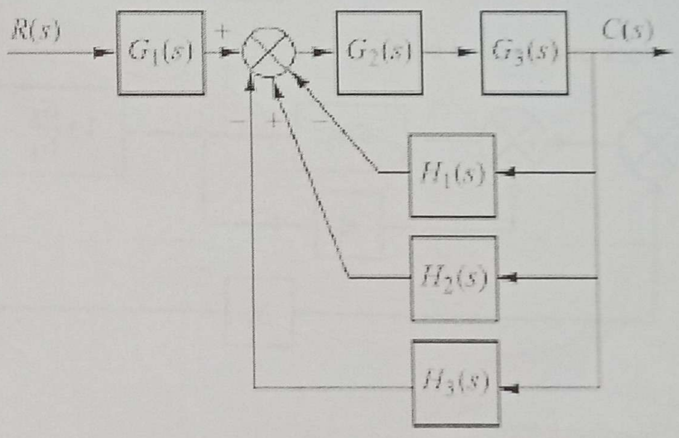


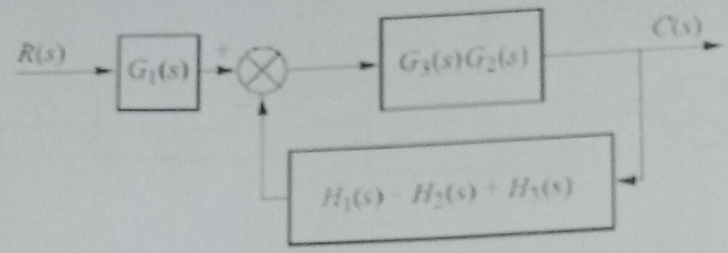


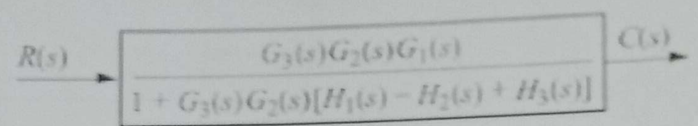
Transfer function

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 (G_3 + G_4)}{1 + G_1 G_2 H_1 - G_1 G_2 (G_3 + G_4) H_2}$$

2. Obtain the transfer function for the system shown in the fig

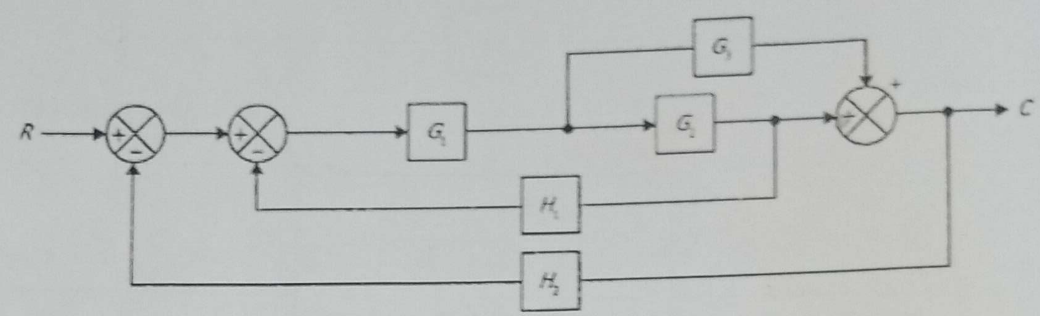




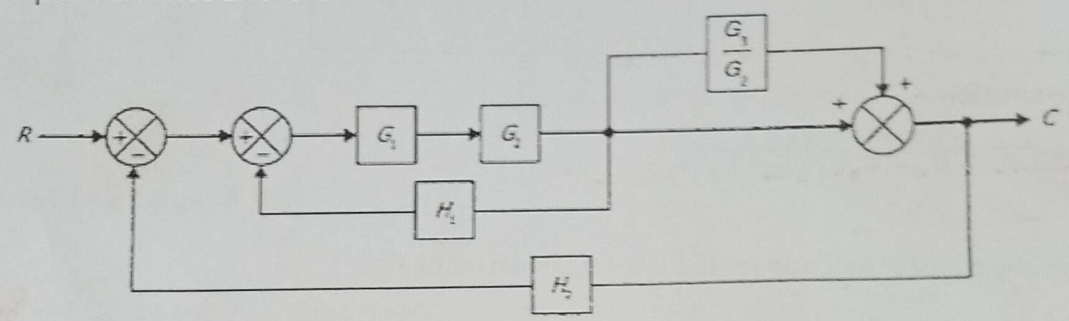


$$\frac{G_3(s)G_2(s)G_1(s)}{1 + G_3(s)G_2(s)[H_1(s) - H_2(s) + H_3(s)]}$$

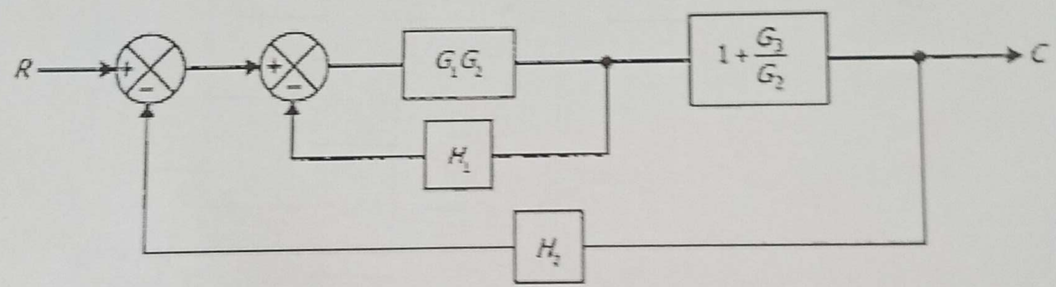
3. Obtain the transfer function C/R for the block diagram shown in the fig



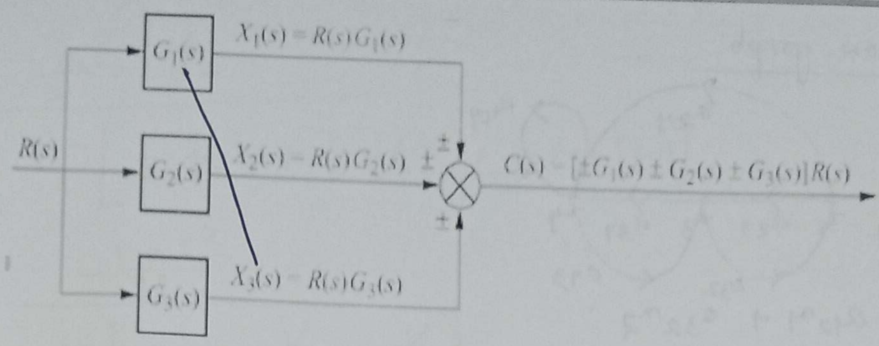
The take-off point is shifted after the block G2



Reducing the cascade block and parallel block

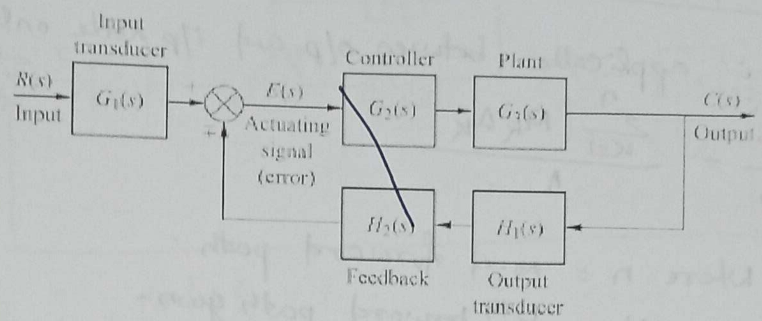


Replacing the internal feedback loop



Parallel Subsystems

Feedback Control System



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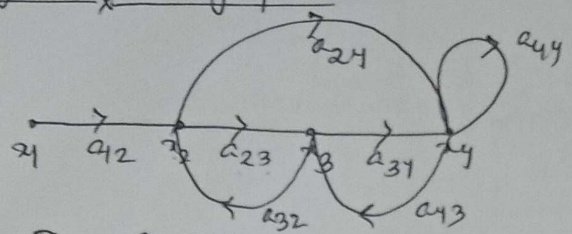
Signal flow graph



$x_2 = a_{12}x_1 + a_{32}x_2$.

$x_3 = a_{23}x_2 + a_{43}x_4$

$x_4 = a_{24}x_2 + a_{44}x_4 + a_{34}x_3$

Mason's gain formula

It is applicable between o/p and i/p node only

$$TF = \frac{o/p}{i/p} = \frac{\sum_{k=1}^{n} M_K \Delta_K}{\Delta}.$$

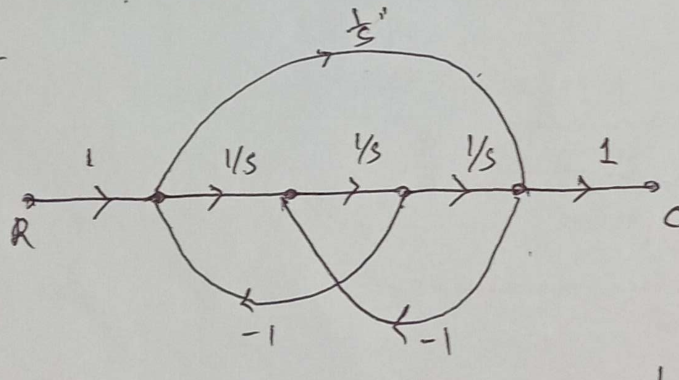
Where n = no. of forward path.

M_K = k^{th} forward path gain.

Δ_K = The value of Δ which not touching kth forward path.

$\Delta = 1 -$ (Sum of loop gains) + (Sum of gain product of two non touching loop) − Sum of gain Product of 3 nontouching loop +

Q/-



Find $\frac{C}{R} = ?$

No. of forward path = $1\frac{1}{s}1$, $1\frac{1}{s}\frac{1}{s}\frac{1}{s}1$.

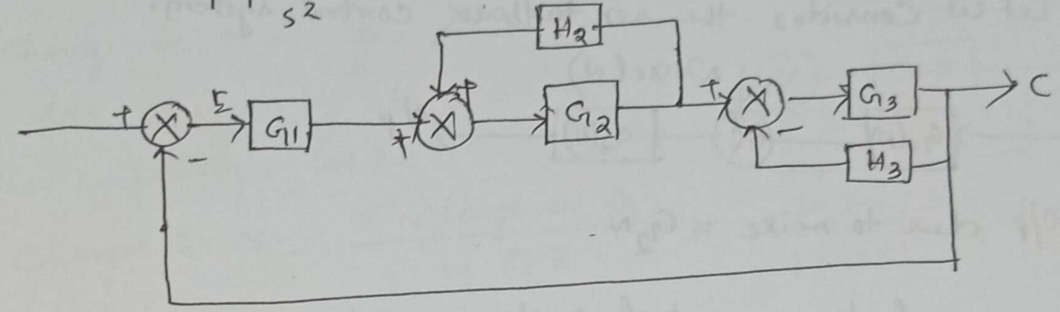
No. of loop = $\left(\frac{1}{s}\frac{1}{s}-1\right)$, $\left(\frac{1}{s}\frac{1}{s}-1\right)$, $\left(\frac{1}{s}-1\frac{1}{s}-1\right)$

$$\frac{C}{R} = \frac{\frac{1}{s} + \frac{1}{s^3}}{1 - \left[\frac{1}{s}\cdot\frac{1}{s}(-1) + \frac{1}{s}\frac{1}{s}(-1) + \frac{1}{s}(-1)\frac{1}{s}(-1)\right]}$$

$$= \frac{\frac{1+s^2}{-s^3}}{1 - \left[-\frac{1}{s^2} - \frac{1}{s^2} + \frac{1}{s^2}\right]}$$

$$= \frac{\frac{1+s^3}{s^3}}{1 + \frac{1}{s^2}} = \frac{1+s^2}{s^3} \times \frac{s^3}{s^2+1} = \frac{s^2}{s^3} = \frac{1}{s} \quad \text{Ans}$$

Q/-



Find $\frac{C}{R}$?, find $\frac{E}{R} = ?$

Ans

No. of forward path = 1 → $G_1 G_2 G_3$

No. of loop = 3 → $G_1 G_2 G_3(-1)$, $G_3 H_3(-1)$, $G_2 H_2$

No. of 2 non touching loop = $G_2 H_2$, $G_3 H_3(-1)$

$$\frac{C}{R} = \frac{G_1 G_2 G_3}{1 - \{-G_1 G_2 G_3 + G_3 H_3(-1) + G_2 H_2\} + \{G_2 H_2 \cdot (-G_3 H_3)\}}$$

$$\frac{E}{R} = \frac{1(1 - G_2 H_2 - G_3 H_3)}{1 - (G_1 G_2 G_3 + G_3 H_2(-1) + G_2 H_2) + G_2 H_2 - G_3 H_3} \quad \text{Ans}$$

Effects of feedback :

(1) Effect of feedback on overall gain :

→ For +ve feedback, gain increases.

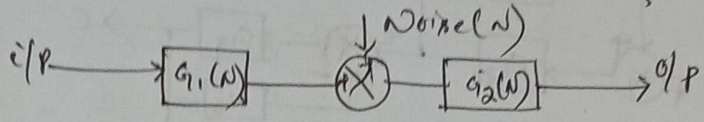
→ For -ve feedback, gain decreases.

(2) Effect of feedback on stability :

If bounded output is obtained from every bounded input then system is said be stable.

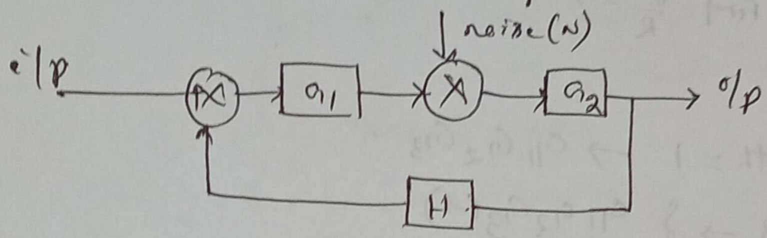
(3) Effect of feedback on noise :

Let us consider the non feedback control system.



∴ o/p due to noise $= G_2 N$

Let us consider feedback control system.



o/p due to noise $= \dfrac{G_2 N}{1 + G_1 G_2 N}$.

So noise is reduced. Further reducing the noise increasing the value of G_1.

Effect of feedback on sensitivity :

Let $M = \dfrac{C}{R} = \dfrac{G}{1+GH}$.

Sensitivity of 'M' to G,

$$S_G^M = \frac{\%\text{ change in } M}{\%\text{ change in } G} = \frac{\frac{\delta M}{M} \times 100\%}{\frac{\delta G}{G} \times 100\%} = \frac{\delta M}{\delta G} \times \frac{G}{M}.$$

$$S_G^M = \frac{d}{dG}(M) \times \frac{G}{M}$$

$$= \frac{d}{dG}\left(\frac{G}{1+GH}\right) \times \frac{G}{M} = \frac{(1+GH) \times 1 - G(H)}{(1+GH)^2} \times \frac{G}{\frac{G}{1+GH}}$$

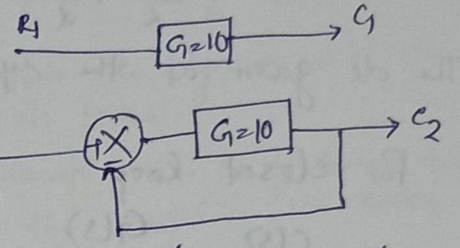
$$\Rightarrow \frac{(1+GH) - G(H)}{(1+GH)^2} \times (1+GH) = \frac{1}{1+GH}.$$

$$S_G^M = \frac{1}{1+GH}.$$

$S_G^M < 1$,

Feedback control system reduce the variation at o/p due to change in G.

Q1- G changes by 10%, then G and C_2?



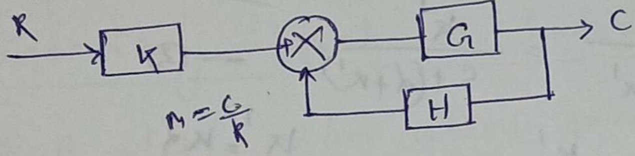
% Change in $M = \frac{C_2}{R}$

G also change by =10 %.

so sensitivity = 100%.

% Change in $M = \frac{\%\ \text{change in}\ G}{1+GH} = \frac{10\%}{1+10\times1} = 1\%.$

Q/-



$M = \frac{C}{R}$

$$M = \frac{C}{R} = \frac{KG}{1+GH}.$$

$$S_K^M = \frac{dM}{dK} \cdot \frac{K}{M} = \frac{d}{dK}\left(\frac{KG}{1+GH}\right) \times \frac{K}{M}.$$

$$= \frac{(1+GH)G}{1+GH^2} \times \frac{K}{\frac{KG}{1+GH}} = 0$$

$$S_H^M = \frac{dM}{dH} \times \frac{H}{M} = \frac{d}{dH}\left(\frac{KG}{1+GH}\right) \times \frac{H}{M} = \frac{(1+GH)\times 0 - (KG)G}{(1+GH)^2} \times \frac{H}{\frac{KG}{1+GH}}$$

$$= \frac{-KG^2}{(1+GH)^2} \cdot \frac{H(1+GH)}{KG} = -\frac{GH}{1+GH}.$$

Note → Sensitivity to feedback path parameter changes the forward path parameter changes.

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If we apply input $r(t) = \delta(t)$ (impulse function).

then $R(s) = 1$.

$$c(t) = \mathcal{L}^{-1}\left(\frac{K'}{s+\alpha}\right) = K' e^{-\alpha t} = K' e^{-t/\tau} \quad \text{(open loop)}.$$

$$c(t) = \mathcal{L}^{-1} \frac{K'}{s+\alpha(1+K)} = K' e^{-\alpha(1+K)t}$$

$$= K' e^{-t/\tau_c} \quad \text{(closed loop)}.$$

→ If $\tau_c = \frac{\tau}{1+K}$ means response decays much faster, which means that the speed of system's response increases by $(K+1)$ as compared to open loop system.

→ It is concluded that feedback controls the dynamics of the system by adjusting the location of its poles.

→ The closed loop system has a bandwidth $(1+K)$ times the bandwidth of the open-loop system this implies increased speed to response.

Linearizing effect of feedback

In a static system various gains are independent of time. We shall assume that the forward block function is non linear expressed as

$$c = f(e) = e^2 \text{ ; square law function.}$$

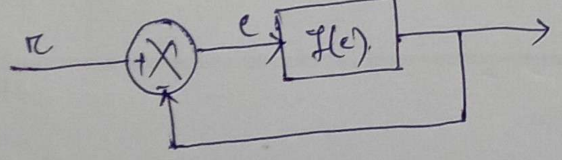
When the feedback loop is open

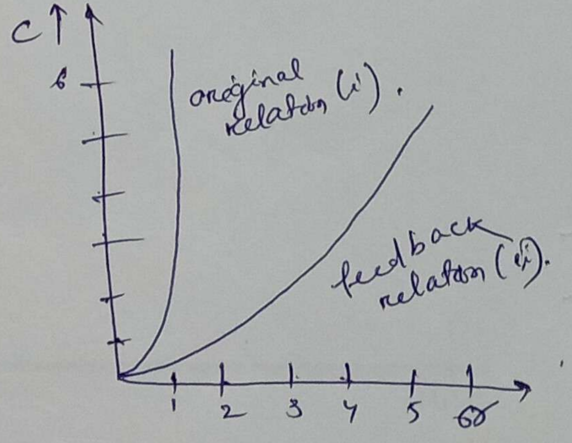
$$e = r \Rightarrow c = r^2$$

For a closed loop system,

$$e = r - c.$$

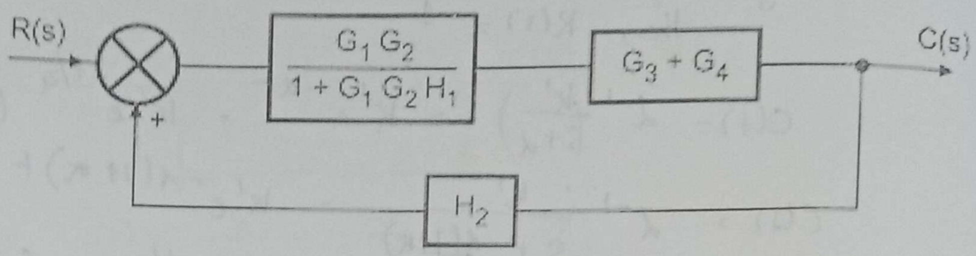
and $$c = f(e) = (r-c)^2$$

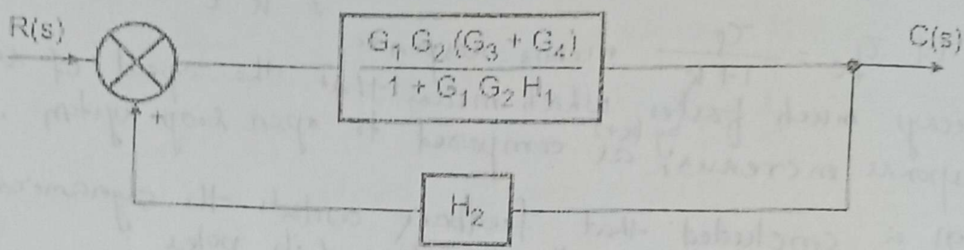


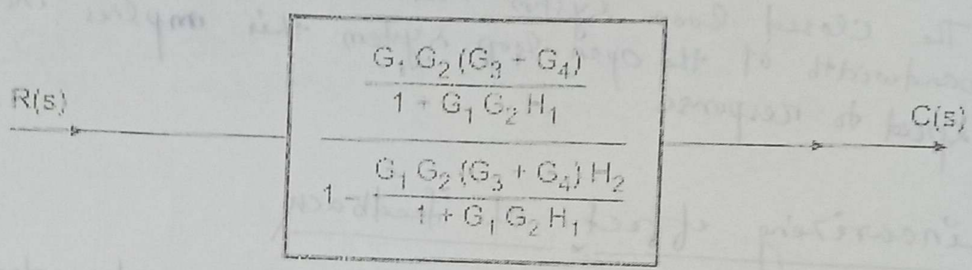


The graph (i) and (ii) that the i/p - o/p relation is approximately linear over a much wider range for the closed loop system compared to its open loop behaviour.

Reduce minor feedback loop of G1, G2 and H1



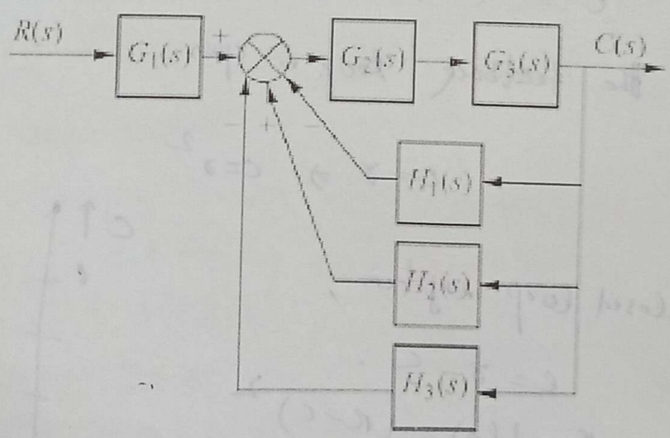




Transfer function

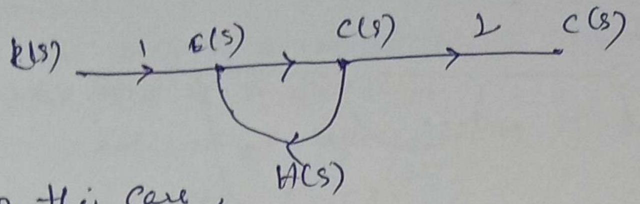
$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 (G_3 + G_4)}{1 + G_1 G_2 H_1 - G_1 G_2 (G_3 + G_4) H_2}$$

2. Obtain the transfer function for the system shown in the fig



Regenerative feedback

In regenerative feedback, the output is feedback with positive sign.

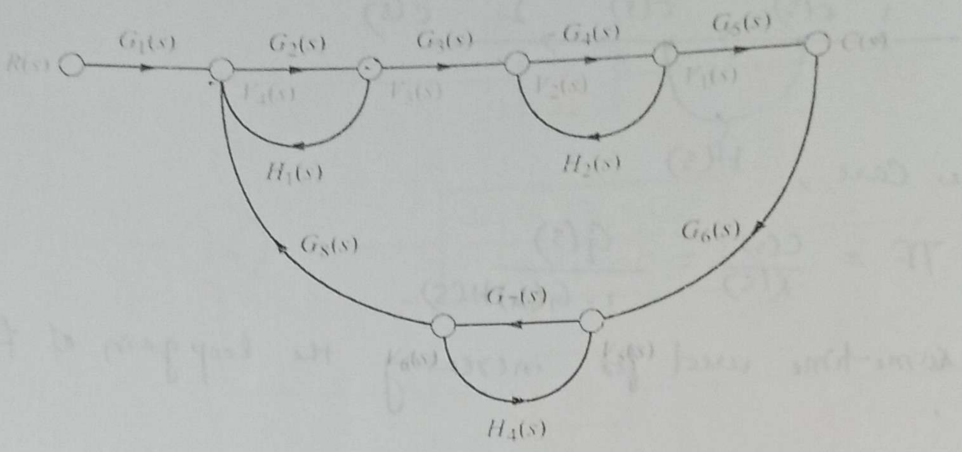


In this case,

$$TF = \frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}.$$

→ It is sometime used for increasing the loop gain of feedback systems.

Problem



Forward path gain: $T_1 = G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$

Closed loop gain

1. $G_2(s)H_1(s)$
2. $G_4(s)H_2(s)$
3. $G_7(s)H_4(s)$
4. $G_2(s)G_5(s)G_4(s)G_3(s)G_6(s)G_7(s)G_8(s)$

Non touching loops taken two at a time

5. Loop 1 and loop 2: $G_2(s)H_1(s)G_4(s)H_2(s)$
6. Loop 1 and loop 3: $G_2(s)H_1(s)G_7(s)H_4(s)$
7. Loop 2 and loop 3: $G_4(s)H_2(s)G_7(s)H_4(s)$

Non touching loops taken three at a time

8. Loop 1, 2, 3: $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

Now, $\Delta = 1 - \{(1) + (2) + (3) + (4)\} + \{(5) + (6) + (7)\} - (8)$

Portion of Δ not touching the forward path

$\Delta_1 = 1 - G_7(s)H_4(s)$

Hence,

$$G(s) = \frac{C(s)}{R(s)} = \frac{T_1\Delta_1}{\Delta}$$

$$= \frac{G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)[1 - G_7(s)H_4(s)]}{\Delta}$$

MODULE - 2

Standard test signals

Step signal :-

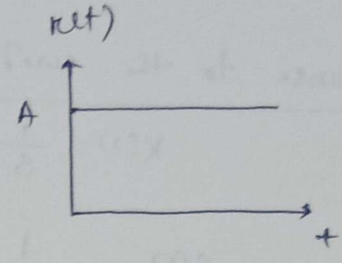
The step is a signal whose value changes from one level to another level A in zero time.

The mathematical representation of the step function is

$$r(t) = A\,u(t)$$

$$u(t) = 1 \;;\; t > 0.$$

$$= 0 \;;\; t < 0.$$



Then $R(s) = \mathcal{L}[A\,u(t)]$

$$= \frac{A}{s}$$

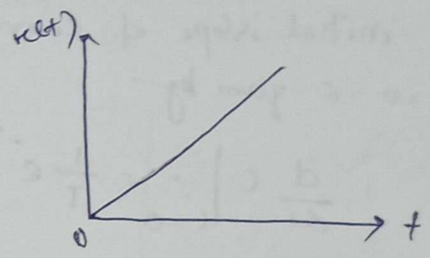
Ramp function

The ramp is a signal which starts at a value of zero and increases linearly with time.

Mathematically,

$$r(t) = At \;;\; t > 0.$$

$$= 0 \;;\; t < 0.$$



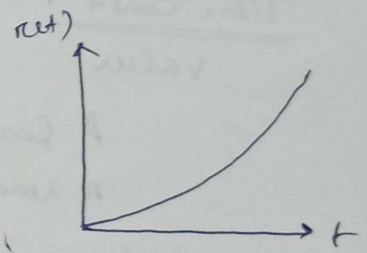
Then $R(s) = \frac{A}{s^2}$.

parabolic signal

The mathematical representation of this signal is

$$r(t) = \frac{At^2}{2} \;;\; t > 0$$

$$= 0 \;;\; t < 0.$$



Then $R(s) = \frac{A}{s^3}$.

Impulse signal (or) δ - function.

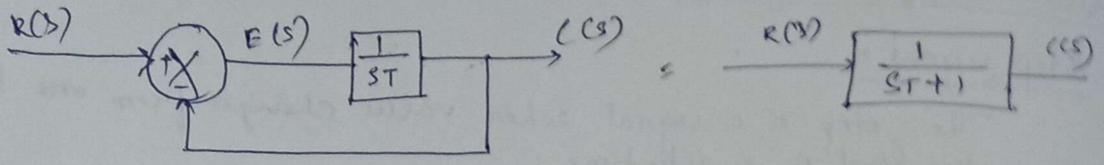
A unit impulse is defined as a signal which has zero value everywhere except $t=0$, where its magnitude is infinite.

$$\delta(t) = 0 \;;\; t \neq 0.$$

$$\int_{-\epsilon}^{+\epsilon} \delta(t)\,dt = 1.$$

Where ϵ tends to zero.

Time response of first order system



$$TF = \frac{C(s)}{R(s)} = \frac{1}{Ts+1}$$

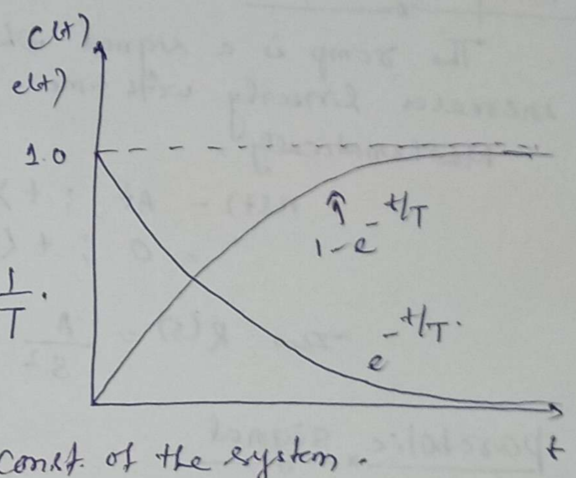
Response to the unit step i/p

$$R(s) = \frac{1}{s}$$

$$C(s) = \frac{1}{s(Ts+1)} = \frac{1}{s} - \frac{T}{Ts+1}$$

Inverse L.T

$$c(t) = 1 - e^{-t/T}$$



The initial slope of curve at $t=0$ is given by

$$\frac{d}{dt} c \bigg|_{t=0} = \frac{1}{T} e^{-t/T} \bigg|_{t=0} = \frac{1}{T}$$

T is known as the time const. of the system.

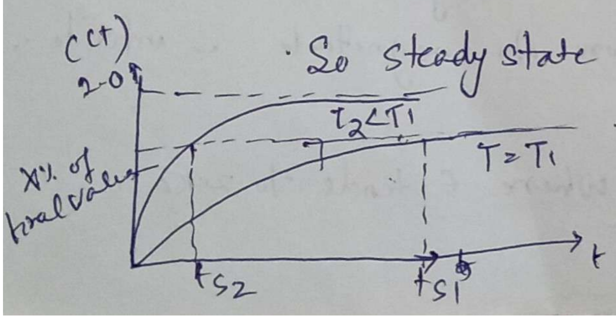
Time const : How fast the system tends to reach final value.

A large time const, A sluggish system.
A small timeconst, fast response.

Consider now,

$$\text{error } e(t) = r(t) - c(t) = e^{-t/T}$$

So steady state error $e_{ss} = \lim_{t \to \infty} e(t) = 0$.



Response to the unit Ramp i/p

$$R(s) = \frac{1}{s^2}$$

$$C(s) = \frac{1}{s^2(Ts+1)} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

Taking inverse laplace transform,

$$c(t) = t - T\left(1 - e^{-t/T}\right).$$

The error signal is

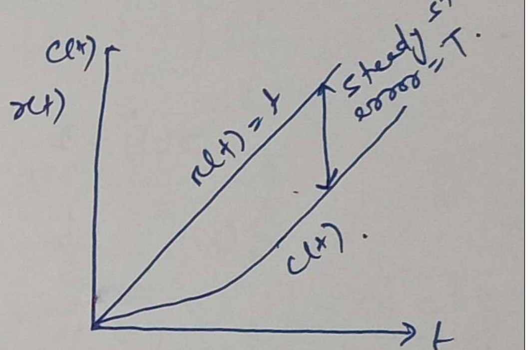
$$e(t) = r(t) - c(t)$$

$$= T\left(1 - e^{-t/T}\right).$$

Steady state error is given by

$$e_{ss} = \lim_{t \to \infty} e(t) = T.$$

Reducing the system time constant therefore not only improves its speed of response but also reduces its steady state error to a ramp i/p.



By applying final value theorem,

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s).$$

$$= \lim_{s \to 0} s\left[R(s) - C(s)\right]$$

$$= \lim_{s \to 0} s\left[\frac{1}{s^2} - \frac{1}{s^2(Ts+1)}\right] = T.$$

Response to the unit impulse function

$$C(s) = R(s)\frac{1}{sT+1} = 1\,\frac{1}{sT+1} = \frac{1}{sT+1}.$$

Taking inverse laplace transform.

$$\mathcal{L}^{-1}C(s) = \mathcal{L}^{-1}\left(\frac{1}{sT+1}\right) = \mathcal{L}^{-1}\left\{\frac{1}{T}\cdot\frac{1}{\left(s+\frac{1}{T}\right)}\right\}$$

$$c(t) = \frac{1}{T}e^{-t/T}.$$

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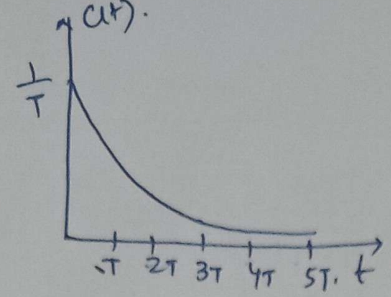
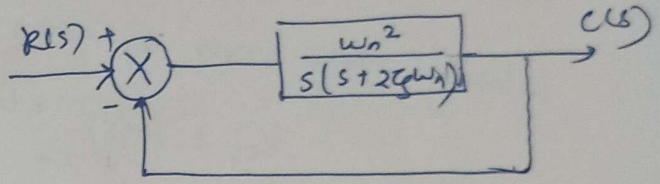


i/p func^n	Time response	Remark
Unit ramp	$c(t) = t - T + Te^{-t/T}$	Differentiate ↓ / ↑ integrate
Unit step	$C(t) = 1 - e^{-t/T}$	
Unit impulse	$C(t) = \frac{1}{T} e^{-t/T}$	

Time response of second order control system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



For unit step i/p →

$$C(s) = R(s) \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

$$r(t) = 1, \quad R(s) = \frac{1}{s}.$$

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

$$\left[\begin{aligned} & s^2 + 2\zeta\omega_n s + \omega_n^2 \\ &= s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2 \\ &\Rightarrow (s + \zeta\omega_n)^2 + \omega_n^2\{1 - \zeta^2\} \end{aligned} \right]$$

~~$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$~~

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

Put $\omega_d = \omega_n\sqrt{1 - \zeta^2}$.

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \quad \text{—— (1)}$$

eq^n ① can be written as

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

Taking inverse Laplace

$$\mathcal{L}^{-1}C(s) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right]$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\cdot\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\cdot e^{-\zeta\omega_n t}\cdot\sin\omega_d t \quad — ③$$

Since $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[(\sqrt{1-\zeta^2})\cos\omega_d t + \zeta\sin\omega_d t\right]$$

Put $\sin\phi = \sqrt{1-\zeta^2}$, $\cos\phi = \zeta$.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\phi\cdot\cos\omega_d t + \cos\phi\,\sin\omega_d t\right)$$

$$\boxed{c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)}$$

Since $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left((\omega_n\sqrt{1-\zeta^2}\,t) + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right) \quad — ②$$

The error is given as.

$$e(t) = r(t) - c(t)$$

$r(t) = 1$

$$e(t) = 1 - \left\{1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right\}$$

$$= \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)$$

The steady state error r

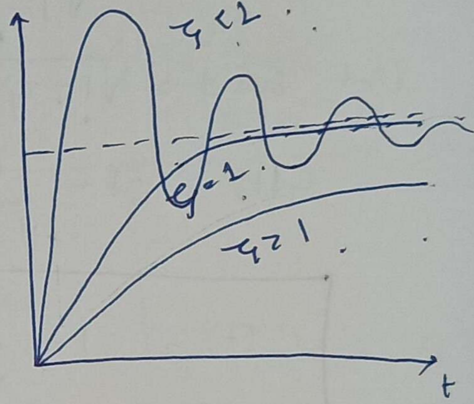
$$e_{ss} = \lim_{t \to \infty} \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\left(1-\zeta^2\right)\right)$$

∴ eqⁿ ② indicate that the value of $\zeta < 1$, the response present exponentially decaying oscillation having freq $\omega_n\sqrt{1-\zeta^2}$ and the time constant of exponential decay is $1/\zeta\omega_n$.

ω_n is called natural frequency of oscillation.

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

ω_d ↓ Damped freq. ; ζ ↳ Damping ratio.



$\zeta\omega_n$ → Damping factor.

For $\zeta < 1$ → Under damping

$\zeta = 1$ → Critical damped.

$\zeta > 1$ → Over damped.

$$\zeta = \frac{\zeta\omega_n}{\omega_n} = \frac{\text{Actual damping}}{\text{Critical damping}}$$

<u>Characteristics equation</u>

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

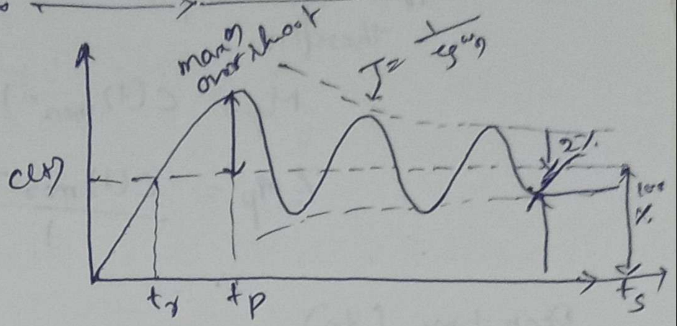
$\Rightarrow s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ ④ → characteristics eqⁿ of second order system.

Roots are

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Transient response specifications of second order control system.

The Rise time t_r



It is the time needed for the response to reach from 10 to 90%. (or) 0 to 100% of the desired value of the o/p at the very ~~fast~~ first instant.

$$c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin\left[\omega_n\left(\sqrt{1-\zeta^2}\right)t + \phi\right].$$

At the first instant when time response reaches 100% of the desired value, i.e $c(t) = 1$ at time t_r.

By substituting $c(t) = 1$,

$$1 = 1 - \frac{e^{-\zeta \omega_n t_r}}{\sqrt{1-\zeta^2}} \sin\left[\omega_n\left(\sqrt{1-\zeta^2}\right)t_r + \phi\right].$$

$$\Rightarrow \frac{e^{-\zeta \omega_n t_r}}{\sqrt{1-\zeta^2}} \sin\left[\omega_n\left(\sqrt{1-\zeta^2}\right)t_r + \phi\right] = 0.$$

$$\frac{e^{-\zeta \omega_n t_r}}{\sqrt{1-\zeta^2}} = \text{finite}.$$

So $$\sin\left[\omega_n\left(\sqrt{1-\zeta^2}\right)t_r + \phi\right] = 0.$$

$$\omega_n\left(\sqrt{1-\zeta^2}\right)t_r + \phi = \pi.$$

$$t_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} \quad \text{--- (3)}$$

where $$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad \text{--- (3)}$$

Maximum Overshoot (M_p).

The maximum positive deviation of the o/p with respect to its desired value is known as maximum overshoot

If the i/p is unit step, the desired output is unity.
therefore,

$$M_p = c(t)_{max} - 1$$

$$\% M_p = \frac{c(t)_{max} - 1}{1} \times 100$$

Peak time (t_p).

The time needed to reach the max^m overshoot is called peak time

The expression for c(t) is

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin[(\omega_n \sqrt{1-\zeta^2})t + \phi]$$

$$\frac{dc(t)}{dt} = -\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \omega_n \sqrt{1-\zeta^2} \cdot \cos(\omega_n(\sqrt{1-\zeta^2})t + \phi) - \frac{(-\zeta\omega_n) e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin[(\omega_n \sqrt{1-\zeta^2})t + \phi]$$

put $\frac{dc(t)}{dt} = 0$.

$$\therefore \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left\{ -\omega_n\sqrt{1-\zeta^2} \cdot \cos[(\omega_n \sqrt{1-\zeta^2})t + \phi] + \zeta\omega_n \sin[(\omega_n\sqrt{1-\zeta^2})t + \phi] \right\} = 0.$$

$$\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} = \text{finite}.$$

$$+\omega_n\sqrt{1-\zeta^2} \cdot \cos[(\omega_n\sqrt{1-\zeta^2})t + \phi] = \zeta\omega_n \sin[(\omega_n\sqrt{1-\zeta^2})t + \phi]$$

Since $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$

$$\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan[\omega_n(\sqrt{1-\zeta^2})t + \phi] \rightarrow (1)$$

Putting the value of ϕ in eq^n (4)

$$\tan\left[\omega_n\left(\sqrt{1-\zeta^2}\right)t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right] = \frac{\sqrt{1-\zeta^2}}{\zeta} \quad -(5).$$

General solution of the above eq^n is

$$\omega_n\sqrt{1-\zeta^2}\,t = n\pi$$

where $n = 0, 1, 2, 3 \dots$

Putting $n = 1$

$$\text{then } t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad -(6)$$

$$* \quad c(t)_{max} = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t_p + \phi\right]$$

$$= 1 - \frac{e^{\frac{-\zeta\omega_n\pi}{\omega_n\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} + \phi\right]$$

$$= 1 - \frac{e^{\frac{-\zeta\omega_n\pi}{\omega_n\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin[\pi + \phi) = 1 - \frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}(-\sin\phi)$$

$$\because \quad \phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

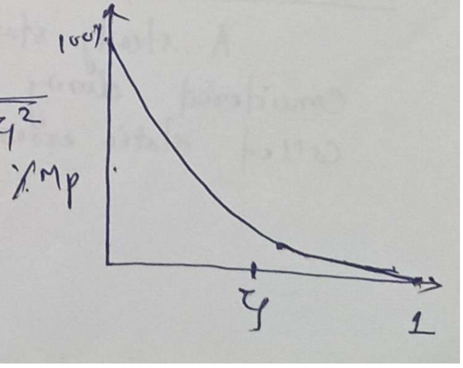
$$\therefore \quad \sin\phi = \sqrt{1-\zeta^2}$$

$$c(t)_{max} = 1 + \frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \cdot \sqrt{1-\zeta^2} \Rightarrow 1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$M_p = c(t)_{max} - 1$$

$$= \left(1 + e^{\frac{-\zeta\pi}{1-\zeta^2}}\right) - 1 = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100.$$



The settling time (t_s).

For an under damped system, output time response decay exponentially with a time const $1/\zeta\omega_n$.

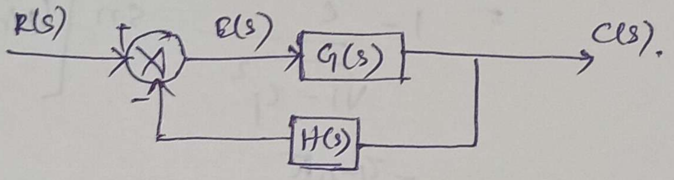
* The time needed to settle down aforesaid oscillation within 2% of desired value of the o/p is known as settling time and denoted as t_s.

$$t_s = \frac{4}{\zeta\omega_{ns}} \quad (\text{for second order system})$$

Steady state error

If the actual o/p of a control system during steady state deviates from the reference input (i.e desired a) the system is said to posses a steady state error.

Consider a closed loop control system,



$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)}.$$

$$E(s) = R(s) \times \frac{1}{1+G(s)H(s)}$$

Applying final value theorem, the steady state error can be determined as follows :-

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s) \times \frac{1}{1+G(s)H(s)}.$$

Static error coefficient

A steady state error for a control system is considered during the steady state period, the error is also called static error.

Static position error coefficient (K_p).

K_p is associated with unit step i/p applied to a closed loop control system.

$$e_{ss} = \lim_{s \to 0} s R(s) \frac{1}{1+G(s)H(s)}.$$

$$R(s) = 1/s.$$

$$e_{ss} = \lim_{s \to 0} s \times \frac{1}{s} \frac{1}{1+G(s)H(s)}$$

$$= \frac{1}{1+\lim_{s \to 0} G(s)H(s)}$$

where $K_p = \lim_{s \to 0} G(s)H(s)$.

$$e_{ss} = \frac{1}{1+K_p}$$

ii) Static velocity error coefficient -

K_v is associated with unit RAMP i/p applied to a closed loop control system

$$e_{ss} = \lim_{s \to 0} s R(s) \frac{1}{1+G(s)H(s)}.$$

$$R(s) = 1/s^2$$

$$e_{ss} = \lim_{s \to 0} s \frac{1}{s^2} \frac{1}{1+G(s)H(s)},$$

$$\Rightarrow \lim_{s \to 0} \frac{1}{s+sG(s)H(s)} = \lim_{s \to 0} \frac{1}{sG(s)H(s)}.$$

put $K_v = \lim_{s \to 0} sG(s)H(s)$.

$$e_{ss} = \frac{1}{K_v}$$

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iii) - Static acceleration error coefficient -
K_a is associated with unit parabolic i/p applied to a closed loop control system.

$$e_{ss} = \lim_{s \to 0} sR(s) \frac{1}{1+G(s)H(s)}$$

$$R(s) = \frac{1}{s^3}$$

$$e_{ss} = \lim_{s \to 0} s \times \frac{1}{s^3} \frac{1}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)H(s)} = \lim_{s \to 0} \frac{1}{s^2 (G(s)H(s))}$$

$$K_a = \lim_{s \to 0} s^2 G(s) H(s).$$

$$\Rightarrow e_{ss} = \frac{1}{K_a}.$$

Types of transfer function and steady state error

① a) Type 0 system with unit step i/p -

$$G(s)H(s) = \frac{K(1+sT_a)(1+sT_b) + \cdots}{(1+sT_1)(1+sT_2) + \cdots}$$

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$= \lim_{s \to 0} \frac{K(1+sT_a)(1+sT_b)\cdots}{(1+sT_1)(1+sT_2)\cdots}$$

$$K_p = K.$$

Since $e_{ss} = \frac{1}{1+K_p}$

$$e_{ss} = \frac{1}{1+K}$$

ⓑ Type 0 system with unit Ramp i/p

$$G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)\cdots}{(1+sT_1)(1+sT_2)\cdots}$$

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} s \frac{K(1+sT_a)(1+sT_b)\cdots}{(1+sT_1)(1+sT_2)\cdots}$$

$k_v = 0$.

Since $e_{ss} = \frac{1}{k_v} = \frac{1}{0} = \infty$.

③ Type 0 system with unit parabolic i/p :-

$$G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)\cdots}{(1+sT_1)(1+sT_2)\cdots}$$

$$k_a = \lim_{s \to 0} s^2 G(s)H(s).$$

$$= \lim_{s \to 0} s^2 \frac{K(1+sT_a)(1+sT_b)\cdots}{(1+sT_1)(1+sT_2)\cdots}$$

$K_a = 0$.

$$e_{ss} = \frac{1}{k_a} = \frac{1}{0} = \infty.$$

i/p	L(input)	Type-0 Static error coefficient	Type-0 e_{ss}	Type-1 Static error coefficient	Type-1 e_{ss}	Type-2 Static error coefficient	Type-2 e_{ss}
Step	$\frac{1}{s}$	$k_p = K$	$\frac{1}{1+K}$	$k_p = \infty$	0	$k_p = \infty$	0
Ramp	$\frac{1}{s^2}$	$k_v = 0$	∞	$k_v = K$	$\frac{1}{K}$	$k_v = \infty$	0
parabolic	$\frac{1}{s^3}$	$K_a = 0$	∞	$K_a = 0$	∞	$k_v = K$	$\frac{1}{K}$

Stability

A linear time invariant system is said to be stable, if satisfies the following condition.

→ Even after excitation by bounded i/p, output must be bounded.

→ In the absence of i/p, output must be zero irrespective of initial condition.

Absolute stability

The stability of the system with respect to variation of parameters is known as absolute stability.

→ For a system to be absolutely stable, all the roots of the Ch. eqn should lie on the left half of s-plane.

Relative stability

It measures the degree of stability.

→ It is a quantitative measure of how fast the system transient die out in a system.

→ Relative stability is a measure of how close the system is to instability.

→ Relative stability is measured in terms of gain ~~and~~ margin and phase margin.

Closed Loop Poles and Stability

Nature of Closed Loop Poles	Locations of Closed Loop Poles in s-plane	Step Response	Stability Condition
Real, negative *i.e.*, in LHS of *s*-plane.	$j\omega$, $-a_2$, $-a_1$, 0, σ	$c(t)$, t — Pure Exponential	Absolutely stable
Complex conjugate with negative real part *i.e.*, in LHS of *s*-plane.	$j\omega$, $j\omega_1$, $-a_1$, 0, $-j\omega_2$, σ	$c(t)$, t — Damped oscillations	Absolutely stable
Real, positive *i.e.*, in RHS of *s*-plane (any one closed loop pole in right half irrespective of number of poles in left half of *s*-plane).	$j\omega$, $+a_1$, σ	$c(t)$, ∞, t — Exponential but increasing towards ∞	Unstable
Complex conjugate with positive real part *i.e.*, in RHS of *s*-plane.	$j\omega$, $j\omega_1$, $-j\omega_2$, σ	$c(t)$, t — Oscillations with increasing amplitude	Unstable
Non-repeated pair on imaginary axis without any pole in RHS of *s*-plane.	$j\omega$, $j\omega_1$, $-j\omega_1$, σ	$c(t)$, t — Frequency of oscillations = ω_1	Marginally or critically stable
Two non-repeated pairs on imaginary axis	$j\omega$, $j\omega_2$, $j\omega_1$, $-j\omega_1$, $-j\omega_2$, σ	$c(t)$, t — Sustained oscillations with two frequency components ω_1 and ω_2	Marginally or critically stable
Repeated pair on imaginary axis without any pole in RHS of *s*-plane.	$j\omega$, $j\omega_1$, $-j\omega_1$, σ	$c(t)$, t — Oscillations of increasing amplitude	Unstable

Routh - Hurwitz Criterion

In order to determine the existence of a root having positive real parts for a polynomial eqⁿ given by

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0.$$

$$\begin{matrix} a_0 & a_2 & a_4 & a_6 & \dots \\ a_1 & a_3 & a_5 & \dots \end{matrix}$$

For 6 the order polynomial,

$$\begin{matrix} s^6 & a_0 & a_2 & a_4 & a_6. \\ s^5 & a_1 & a_3 & a_5 & 0 \\ s^4 & \frac{a_1a_2 - a_3a_0}{a_1} & \frac{a_1a_4 - a_0a_5}{a_1} & \frac{a_1a_6 - a_0a_7}{a_1} & \frac{a_1a_8}{a_1} \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{matrix}$$

Ex.

Apply Routh criterion to determine the stability of the system having the Ch. eqⁿ as

$$s^3 + 4\times10^2 s^2 + 5\times10^4 s + 2\times10^6 = 0.$$

Ans

Substituting $s = \alpha p$.
where $\alpha = 10^2$.

$$((10)^2 p)^3 + 4\times10^2(10^2 p)^2 + 5\times10^4(10^2 p) + 2\times10^6 = 0$$

$$\Rightarrow \alpha^3 p^3 + 4\alpha\times\alpha^2\times p^2 + 5\times\alpha^2(\alpha p) + 2\times\alpha^3 = 0$$

$$\Rightarrow \alpha^3(p^3 + 4p^2 + 5p + 2) = 0$$

$$\Rightarrow p^3 + 4p^2 + 5p + 2 = 0.$$

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p^3	1	5	
p^2	4	2	
p^1	4.5	0	
p_0	2	0	

All the 1st column elements are of the same sign, therefore system is stable.

Ex

Determine the stability of the system whose overall transfer function is given below,

$$\frac{C(s)}{R(s)} = \frac{2s+5}{s^5 + 1.5s^4 + 2s^3 + 4s^2 + 5s + 10}$$

Ans The ch. eqn is

$$s^5 + 1.5s^4 + 2s^3 + 4s^2 + 5s + 10 = 0.$$

s^5	1	2	5
s^4	1.5	4	10
s^3	-0.66	-1.66	0
s^2	0.227	10	0
s^1	27.4	0	0
s^0	10	0	0

There are two changes of sign of the elements in the first column that means the system has two roots with positive real parts and as such the system is unstable.

Ex Determine the stability of the closed loop system whose characteristics eqn:

$$s^5 + s^4 + 2s^3 + 2s^2 + 11s + 10 = 0.$$

The Routh array is formed below

s^5	1	2	11
s^4	1	2	10
s^3	0 (ϵ)	1	0
s^2	$\lim_{\epsilon \to 0} \frac{\epsilon \times 2 - 1 \times 1}{\epsilon} = -\infty$	10	0
s^1	$\lim_{\epsilon \to 0} \frac{1 - 10\epsilon^2}{2\epsilon - 1} = 1$	0	0
s^0	10		

When first column of 1st row is zero then substitute ϵ then solve the table

There are two sign changes means the system is unstable.

Ex Determine the stability of a system having following ch. eqn.

$$s^6 + s^5 + 5s^4 + 3s^3 + 2s^2 - 4s - 8 = 0.$$

Ans

s^6	1	5	2	-8
s^5	1	3	-4	0
s^4	2	6	-8	0 → Auxillary eqn.
s^3	0	0	0	0
s^2				

$A(s) = 2s^4 + 6s^2 - 8$

$\frac{d}{ds} A(s) = 8s^3 + 12s + 0$

s^6	1	5	2	-8
s^5	1	3	-4	0
s^4	2	6	-8	0
s^3	8	12	0	0
s^2	3	-8	0	0
s^1	100/3	0	0	0
s^0	-8	0	0	0

As there is one sign changes in the first column, the system is unstable.

Note

→ Equal real roots with opposite sign. As one roots is +ve, the system is unstable.

→ A pair of conjugate root on imaginary axis. This gives marginal stability. There is no sign changes in the 1st column.

→ There is no changes in the 1st column of Routh's tabulation but existence of two rows having zero elements make the system unstable.

Problem

The open loop transfer function of a unity feed back control system is given by

$$G(s) = \frac{K}{s(sT_1+1)(sT_2+1)}$$

Determine the value of K in term of T_1 and T_2 for the system to be stable.

Ans

$$1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{K}{s(sT_1+1)(sT_2+1)} = 0$$

$$\Rightarrow s(sT_1+1)(sT_2+1) + K = 0$$

$$\Rightarrow (s^2T_1 + s)(sT_2+1) + K = 0$$

$$\Rightarrow s^3T_1T_2 + s^2T_1 + s^2T_2 + s + K = 0$$

$$\Rightarrow s^3T_1T_2 + s^2(T_1+T_2) + s + K = 0$$

s^3	T_1T_2	1
s^2	T_1+T_2	K
s^1	$\frac{T_1+T_2-KT_1T_2}{T_1+T_2}$	0
s^0	K	

For the system to be stable (i) $K > 0$:

$$\frac{T_1+T_2-KT_1T_2}{T_1+T_2} > 0 \Rightarrow T_1+T_2-KT_1T_2 > 0$$

$$\Rightarrow KT_1T_2 < T_1+T_2 \Rightarrow K < \frac{T_1+T_2}{T_1T_2}$$